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Cohomology of Coherent Sheaves and Series of Supernatural Bundles

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Abstract

We show that the cohomology table of any coherent sheaf on projective space is a convergent—but possibly infinite—sum of positive real multiples of the cohomology tables of what we call *supernatural* sheaves.

Introduction

Let $\mathbb{K}$ be a field, and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{K}}$. The *cohomology table* of $\mathcal{F}$ is the collection of numbers

$$\gamma(\mathcal{F}) = (\gamma_{i,d}) \text{ with } \gamma_{i,d} = \dim H^i(\mathbb{P}^n, \mathcal{F}(d)),$$

which we think of as an element of the real vector space $\prod_{d=-\infty}^{\infty} \mathbb{R}^{n+1}$.

In Eisenbud-Schreyer [2009] we characterized the cohomology tables of vector bundles on $\mathbb{P}^n$ (up to a positive rational multiple) as the finite positive rational linear combinations of cohomology tables of *supernatural* bundles, which we described explicitly. In this paper we treat the cohomology tables of all coherent sheaves. These are given by infinite sums:

Theorem 0.1. *The cohomology table of any coherent sheaf on $\mathbb{P}^n$ can be written as a convergent series, with positive real coefficients, of cohomology tables of supernatural bundles supported on linear subspaces.*

We actually prove a more precise result, which includes a uniqueness statement. To state it we recall some ideas from Eisenbud-Schreyer [2009].

A sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has *supernatural cohomology* if, for each integer d, the cohomology $H^i(\mathcal{F}(d))$ is nonzero for at most one value of i and, in addition, the

Hilbert polynomial $d \mapsto \chi(\mathcal{F}(d))$ has distinct integral roots. We define the *root sequence* of a supernatural sheaf $\mathcal{F}$ to be the sequence of roots of the Hilbert polynomial, written in decreasing order, $z_1 > \cdots > z_s$ where s is the dimension of the support of $\mathcal{F}$. It will be convenient to put $z_0 = \infty$ and $z_{s+1} = z_{s+2} \ldots = -\infty$.

The Hilbert polynomial and the cohomology table of a supernatural sheaf $\mathcal{F}$ are determined by the root sequence $(z_1, \ldots, z_s)$ and the degree of $\mathcal{F}$ as follows. It is immediate that

$$\chi(\mathcal{F}(d)) = \frac{\deg \mathcal{F}}{s!} \prod_{i=1}^{s} (d - z_i).$$

By Theorem 6.4 of our [2009],

$$h^j \mathcal{F}(d) = \begin{cases} \frac{\deg \mathcal{F}}{s!} \prod_{i=1}^{s} \mid d - z_i \mid & \text{if } z_j > d > z_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 6.1 of that paper, there exists a supernatural sheaf of dimension s and degree $s!$ with any given root sequence $z = (z_1 > \cdots > z_s)$. It may be taken to be a vector bundle on a linear subspace $\mathbb{P}^s \subset \mathbb{P}^n$. We denote its cohomology table by γ^z. Thus γ^z is the cohomology table of a vector bundle if and only if $z_n > -\infty$.

We partially order the root sequences termwise, setting $z \geq z'$ when

$$z_1 \geq z'_1, \ldots, z_n \geq z'_n.$$

By a *chain* we mean a totally ordered set. If Z is an infinite sequence of root sequences, $(q_z)_{z \in Z}$ a sequence of numbers, and γ is a cohomology table, we write $\gamma = \sum_{z \in Z} q_z \gamma^z$, to mean that each entry $\sum_{z \in Z} q_z \gamma^z_{i,d}$ converges to $\gamma_{i,d}$.

With these preparations we can state the precise version of our main result. Recall that a sheaf is said to be *purely s-dimensional* if all its associated subvarieties have dimension exactly s.

Theorem 0.2. *Let $\gamma(\mathcal{F})$ be the cohomology table of a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. There is a chain of zero-sequences Z and positive real numbers q_z such that*

$$\gamma(\mathcal{F}) = \sum_{z \in Z} q_z \gamma^z.$$

Both Z and the numbers q_z are uniquely determined by this condition. The coefficients q_z corresponding to cohomology tables γ^z of dimension $\dim \mathcal{F}$ are rational numbers. If $\mathcal{F}$ is purely s-dimensional, then all the γ^z are cohomology tables of vector bundles on $\mathbb{P}^s$ and all the q_z are rational.

We do not know whether all the numbers q_z are rational, nor whether, if all the γ^z are cohomology tables of vector bundles, the sheaf $\mathcal{F}$ is necessarily torsion-free.

When we want to display (parts) of a cohomology table we use the convention

$$\begin{array}{cccccc|c}
\cdots & \gamma_{n,-n-1} & \gamma_{n,-n} & \gamma_{n,-n+1} & \cdots & & n \\
 & \vdots & \vdots & \vdots & & & \vdots \\
\cdots & \gamma_{1,-2} & \gamma_{1,-1} & \gamma_{1,0} & \cdots & & 1 \\
\cdots & \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \cdots & & 0 \\
\hline
\cdots & -1 & 0 & 1 & \cdots & & d\backslash i
\end{array}$$

We make this choice of indexing so that the cohomology table of a coherent sheaf $\mathcal{F}$ coincides with the Betti table of the *Tate resolution* of $\mathcal{F}$. This is a minimal, doubly infinite, exact free complex over the exterior algebra on $n+1$ generators that is associated to $\mathcal{F}$ by the Bernstein-Gel'fand-Gel'fand correspondence. It is studied in Eisenbud-Fløystad-Schreyer [2003] and Eisenbud-Schreyer [2003]. For consistency with the notation of those papers, we number the rows from the bottom and the columns from left to right as in the table above.

Example 0.3. The ideal sheaf $\mathcal{I}_p$ of a point in $\mathbb{P}^2$ has the cohomology table

$$\begin{array}{ccccccccccc|c}
\cdots & 10 & 6 & 3 & 1 & & & & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & & & & & & 1 \\
 & & & & & & 2 & 5 & 9 & 14 & \cdots & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & d\backslash i
\end{array}$$

where we drop the zero entries to make the shape more visible. The expression in Theorem 0.2 is

$$\gamma(\mathcal{I}_p) = \sum_{k=2}^{\infty} q_{(0,-k)} \gamma^{(0,-k)}$$

where

$$q_{(0,-k)} = \frac{2}{(k-1)k(k+1)}.$$

In particular

$$\sum_{k=2}^{\infty} \frac{2d(d+k)}{(k-1)k(k+1)} = \binom{d+2}{2} - 1$$

holds for any $d \geq 1$,

$$\sum_{k=-d+1}^{\infty} \frac{2d(d+k)}{(k-1)k(k+1)} = -1$$

for any $d \leq -1$ and

$$\sum_{k=2}^{-d} \frac{2d(d+k)}{(k-1)k(k+1)} = \frac{(d+2)(d+1)}{2}$$

for any $d \leq -2$.

To explain the proof of Theorem 0.2, we introduce a little more terminology. We define the i-th *regularity* of a table $\gamma \in \prod_{d=-\infty}^{\infty} \mathbb{R}^{n+1}$ to be

$$z_i(\gamma) = \inf\{d \mid \gamma_{j,e+j} = 0 \text{ for all } j \geq i,\ e \geq d\}.$$

We refer to $z(\gamma) = (z_1(\gamma), \ldots, z_n(\gamma))$ as the *regularity sequence* of γ. It follows immediately from the definition that $z_1(\gamma) > z_2(\gamma) > \cdots$. Note that $z_1(\gamma(\mathcal{F}))$ coincides with the Castelnuovo-Mumford regularity of the sheaf $\mathcal{F}$. If γ is the cohomology table of a supernatural sheaf $\mathcal{F}$, then it follows from Theorem 6.4 of our [2009] that $z_i(\gamma)$ is the i-th root of the Hilbert polynomial of $\mathcal{F}$.

We define the *support* of a table γ to be the set of indices $\{(i,d) \mid \gamma_{i,d} \neq 0\}$, and the *dimension* of γ to be the maximum i such that $\gamma_{i,d} \neq 0$ for some d, or -1 if all the $\gamma_{i,d}$ are zero. Finally, the *corners* and *corner values* of γ are defined to be the positions

$$(i, z_i(\gamma) + i - 1) \text{ and values } \gamma_{i, z_i(\gamma)+i-1}$$

for each i such that $i \leq \dim \gamma$ and $z_{i+1} < z_i - 1$. The decomposition of Theorem 0.2 is effected by a transfinite "greedy algorithm":

Algorithm 0.4. (Decompose a Cohomology Table)
Input: A cohomology table $\gamma = \gamma(\mathcal{F})$ for some coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$.
Output: A chain of root sequences Z and positive real numbers $(q_z)_{z \in Z}$ such that $\gamma = \sum_{z \in Z} q_z \gamma^z$.

1. Set $Z = \{\}$.
2. Set $i = \dim \gamma$.
3. WHILE $\dim \gamma = i$ DO
   (a) Let z be the regularity sequence of γ, and replace Z by $Z \cup \{z\}$
   (b) Let $q_z > 0$ be largest real number such that the corner values of γ are $\geq$ to the corner values of $q_z \gamma^z$.

(c) Replace γ by $\gamma - q_z\gamma^z$.

4. Replace γ by the limit of the tables produced in step 3c.

5. If $\gamma = 0$ then STOP, else go to Step 2.

Note that Step 2 is executed at most n times, but we may loop through Steps 3a–3c infinitely often for each value of i from n to 1.

Outline of the proof that Algorithm 0.4 succeeds. The crucial difficulty in the proof of Theorem 0.2 is to show that table $\gamma - q_z\gamma^z$ produced each time we pass through Step 3c has non-negative entries, and is sufficiently "like" the cohomology table of a coherent sheaf to allow us to continue. To do this we will define a class of tables closed under the basic operation in Step 3, and under taking limits in an appropriate way. We call these *admissible* tables; they are defined in §2.

The proof that Step 3c produces an admissible table is also given in §2. It rests on an understanding of some functionals that are positive on the cohomology tables of sheaves. Some of these functionals were defined in our paper [2009], and §1 contains a simplified description of them, as well as some others necessary for the present proof.

The dimension s of γ is genuinely reduced each time we return to Step 2: Indeed, some corner value of γ becomes zero in Step 3c, decreasing some z_i. Since z_s remains the smallest of the (finite) z_i, only finitely many steps can occur before z_s is reduced, and thus in the course of the WHILE loop, z_s must be reduced to $-\infty$, so the dimension drops in Step 4, if it has not dropped already in Step 3.

The convergence of the limiting process in Step 4 is dealt with in §3, as are the uniqueness and the special case of a pure-dimensional sheaf. Finally, the necessary positivity is proven in §4, following an idea suggested by Rob Lazarsfeld. $\square$

The following example shows that the decomposition of Theorem 0.2 sometimes mixes the torsion and torsion-free parts of a sheaf, even when the sheaf itself is a direct sum.

Example 0.5. Let $\mathcal{I}$ be the ideal sheaf of a point in $\mathbb{P}^2$, and let L be a line in $\mathbb{P}^2$. Set $\mathcal{F} = \mathcal{I} \oplus \mathcal{O}_L(-4)$. The cohomology table of $\mathcal{F}$ is given by the following diagram, where we have marked the corner values with boxes.

$\cdots$	6	3	$\boxed{1}$							2
$\cdots$	8	7	6	5	3	2	$\boxed{1}$			1
					2	5	9	15	$\cdots$	0
$\cdots$	-3	-2	-1	0	1	2	3	4	$\cdots$	$d\backslash i$

The regularity sequence is $z = (-2, 3)$. The supernatural cohomology table γ^z is

$\cdots$	24	14	$\boxed{6}$							2
				4	6	6	$\boxed{4}$			1
								6	$\cdots$	0
$\cdots$	-3	-2	-1	0	1	2	3	4	$\cdots$	$d\backslash i$

so we see that $q_z = 1/6$. The table $\gamma' := \gamma - q_z\gamma^z$ has the form

$\cdots$	2	$\boxed{2/3}$								2
$\cdots$	8	7	6	$13/3$	2	1	$\boxed{1/3}$			1
					2	5	9	14	$\cdots$	0
$\cdots$	-3	-2	-1	0	1	2	3	4	$\cdots$	$d\backslash i$

The regularity sequence of this table is $z' = (-3, 3)$. This time, the corner that is cancelled in γ' is the one in the middle row, which comes from the torsion sheaf $\mathcal{O}_L(-4)$, rather than from $\mathcal{I}$, and the table $\gamma' - q_{z'}\gamma^{z'}$ looks like

$\cdots$	$14/15$	$\boxed{1/5}$								2
$\cdots$	8	7	$17/3$	$19/5$	$7/5$	$\boxed{7/15}$				1
					2	5	9	$203/15$	$\cdots$	0
$\cdots$	-3	-2	-1	0	1	2	3	4	$\cdots$	$d\backslash i$

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1 Positive Functionals on Cohomology Tables

In this section we will define some functionals—that is, real valued functions—of an array

$$\gamma = (\gamma_{j,d}) \in \prod_{d=-\infty}^{\infty} \mathbb{R}^{n+1}.$$

The key to the proof of the Theorem 0.2 is the Positivity Theorem 1.2 below, stating that certain of these functionals take non-negative values on the cohomology tables of coherent sheaves.

Some of the functionals we need were defined in our [2009], and Theorem 1.2 for those functionals, in the case of the cohomology table of a vector bundle, is a translation of what is there. Here we present a much simpler account of the functionals, that adapts well to the new ones we use. The proof of Theorem 1.2 given in §4.

Define the t-th *partial Euler characteristic of the* d*-th twist* of a table $\gamma \in \prod_{d=-\infty}^{\infty} \mathbb{R}^{n+1}$ to be the functional

$$\chi_d^{\leq t}(\gamma) = \sum_{i=0}^{t} (-1)^i \gamma_{i,d}.$$

When $t = \infty$ (or is simply large enough to be irrelevant) we simply write $\chi_d(\gamma)$ instead of $\chi_d^{\leq t}$. For example, the usual Euler characteristic of a sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is $\chi(\mathcal{F}) = \chi_0^{\leq n}(\gamma(\mathcal{F})) = \chi_0(\gamma(\mathcal{F}))$.

If

$$\begin{aligned} d &= d_0, \ldots, d_{s+1} \in \mathbb{Z}, \\ \psi &= \psi_0, \ldots, \psi_{s+1} \in \mathbb{Z} \cup \{\infty\} \end{aligned}$$

are sequences (which we will call *degrees* and *bounds*, respectively) we set

$$r_i = r_i(d) := \prod_{\substack{0 \leq j < k \leq s+1 \\ j,k \neq i}} (d_k - d_j)$$

and define a functional

$$L(d, \psi)\colon \prod_{d \in \mathbb{Z}} \mathbb{R}^{n+1} \to \mathbb{R}$$

by the formula

$$\begin{aligned} L(d, \psi) &= \sum_{i=0}^{s+1} (-1)^i r_i \chi_{-d_i}^{\leq \psi_i} \\ \gamma &\mapsto \sum_{i=0}^{s+1} (-1)^i r_i \chi_{-d_i}^{\leq \psi_i}(\gamma) = \sum_{i=0}^{s+1} (-1)^i r_i \sum_{j=0}^{\psi_i} (-1)^j \gamma_{j,-d_i}. \end{aligned}$$

We write ∞ for the special sequence of bounds $(\infty, \ldots, \infty)$. The naturalness of the functionals $L(d, \psi)$ is suggested by the following well-known result used for interpolating polynomials, and its specialization to our case.

Lemma 1.1. *Let $d = (d_0, \dots, d_{s+1})$ be any sequence of $s+2$ numbers, and let $r_i = r_i(d)$ as above. If γ is the cohomology table of a coherent sheaf of dimension $\leq s$ (or any table of dimension s such that $d \mapsto \chi_d(\gamma)$ is a polynomial of degree $\leq s$) then $L(d, \infty)(\gamma) = 0$.*

Proof. More generally, if $p(t)$ is any polynomial of degree $\leq s$, then

$$\sum_{i=0}^{s+1} (-1)^i r_i p(d_i) = 0.$$

This follows from the fact that the last column of the $(s+2) \times (s+2)$ matrix

$$\begin{pmatrix} 1 & d_0 & \cdots & d_0^s & p(d_0) \\ 1 & d_1 & \cdots & d_1^s & p(d_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & d_s & \cdots & d_s^s & p(d_s) \\ 1 & d_{s+1} & \cdots & d_{s+1}^s & p(d_{s+1}) \end{pmatrix}$$

is linearly dependent on the others, so the determinant vanishes. The displayed formula is the Laplace expansion of this determinant along the last column. $\square$

We will use the $L(d, \psi)$ with some other special sequences of bounds $\psi = \phi^j$ as well. They are defined as follows: For $j = 1, \dots, s$, we define

$$\phi^j(s) \quad = \quad (\phi_0^j, \dots \phi_{s+1}^j),$$

where

$$\phi_i^j \quad = \quad \begin{cases} i & \text{if } i < j \\ i-1 & \text{if } i = j \\ i-2 & \text{if } i > j, \end{cases}$$

or, less formally,

$$\phi^j(s) = (0, \dots, j-2, j-1, j-1, j-1, j, \dots, s-1).$$

Finally, we set $\phi^0(s) = (-1, 0, \dots, s-2, s-1, s-1)$. Here is our main result on the functionals $L(d, \phi^j(s))$:

Theorem 1.2 (Positivity)**.** *Let d be a degree sequence, $d = (d_0 < \cdots < d_{s+1})$ and let $r = r(d)$. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then, for all $j \geq 1$*

$$L(d, \phi^j(s))(\gamma(\mathcal{F})) \geq 0,$$

and

$$-L(d, \phi^0(s))(\gamma(\mathcal{F})) \geq 0.$$

One may visualize the action of the linear form $L(d, \phi^j(s))$ on a cohomology table γ as the dot product of γ with the table illustrated (for the case $s = 6, j = 2$) in Figure 1.

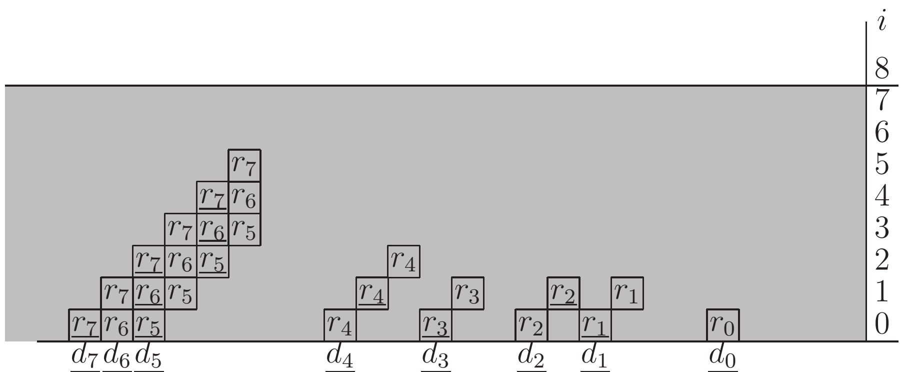

Figure 1: To save space we have denoted $-d_i$ by $\underline{d_i}$ and $-r_i$ by $\underline{r_i}$. The shaded space indicates the positions where a cohomology table of a sheaf of dimension 7 on $\mathbb{P}^8$ could have nonzero values. The functional $L(d, \phi^2(6))$ is the dot product with the table having $\pm r_i$ in the positions shown, which are initial segments of the diagonals numbered $\underline{d_0}, \dots, \underline{d_7}$, and zeros elsewhere.

For the case $j > 0$ the proof, given in §4, follows the same outline as that in our paper [2009]. Using the results of our [2009] and Boij-Söderberg [2008], Theorem 1.2, in the case $j > 0$, is equivalent to Theorem 4.1. We will deduce the case $j = 0$ from the case $j > 0$ by a complicated numerical argument. It would be interesting to give a direct argument for the case $j = 0$ as well.

Here is an example of how Theorem 1 can be applied.

Example 1.3. The Hilbert scheme $Hilb^{2t+2}(\mathbb{P}^3) = H_1 \cup H_2$ has two irreducible components, which we will call H_1 and H_2. The generic point of H_1 corresponds to two skew lines $X \subset \mathbb{P}^3$, while the generic point of H_2 corresponds to $Y = C \cup p \subset \mathbb{P}^3$, where C is a conic and p is a point not in the plane spanned by C. The cohomology table of the ideal sheaf $\mathcal{I}_X$ is

$$\gamma(\mathcal{I}_X) = \begin{array}{ccccccccccc|c}
\cdots & 20 & 10 & 4 & 1 & & & & & & & 3 \\
\cdots & 10 & 8 & 6 & 4 & 2 & & & & & & 2 \\
\cdots & & & & & & 1 & & & & & 1 \\
 & & & & & & & 4 & 12 & 25 & \cdots & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & d\backslash i
\end{array}$$

while that of $\mathcal{I}_Y$ is

$$
\gamma(\mathcal{I}_Y) = \quad
\begin{array}{cccccccccc|c}
\cdots & 20 & 10 & 4 & 1 & & & & & & 3 \\
\cdots & 11 & 9 & 7 & 5 & 3 & 1 & & & & 2 \\
\cdots & 1 & 1 & 1 & 1 & 1 & 1 & & & & 1 \\
 & & & & & & & 4 & 12 & 25 & \cdots & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & d\backslash i
\end{array}
$$

Using Theorem 1 we can show that any integral table "between" these two tables, obtained by replacing the value $h^1\mathcal{I}_Y(d) = 1$ with a zero, and decreasing $h^2\mathcal{I}(d)$ by 1 as well, for some set of values $d < 0$, cannot occur as the cohomology table of any sheaf; and even that no multiple of such a table can occur. For example, no multiple of either the table

$$
T_2 := \quad
\begin{array}{ccccccccccc|c}
\cdots & 20 & 10 & 4 & 1 & & & & & & & 3 \\
\cdots & 10 & 8 & 6 & 4 & 2 & 1 & & & & & 2 \\
\cdots & & & & & 1 & 1 & & & & & 1 \\
 & & & & & & & 4 & 12 & 25 & \cdots & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & d\backslash i
\end{array}
$$

or

$$
T_3 := \quad
\begin{array}{ccccccccccc|c}
\cdots & 20 & 10 & 4 & 1 & & & & & & & 3 \\
\cdots & 10 & 8 & 6 & 4 & 3 & 1 & & & & & 2 \\
\cdots & & & & 1 & 1 & 1 & & & & & 1 \\
 & & & & & & & 4 & 12 & 25 & \cdots & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & \cdots & d\backslash i
\end{array}
$$

can be the cohomology table of a coherent sheaf.

One way to prove such a statement would be to apply Algorithm 0.4, and see that it eventually encounters a table with a negative entry. For instance, in the case of the table T_3, that occurs after 16 steps. But to prove the statement in general, it is easier to appeal directly to Theorem 1.2.

First, consider the functional $L((-1, 1, 2, 3), \phi^2(2))$, which may be written as the dot product with the table

$$
\begin{array}{cccccccc|c}
 & & & & & & & & 3 \\
 & & & & & & & & 2 \\
 & & & 6 & -16 & 12 & & & 1 \\
 & & -6 & 16 & -12 & & 2 & & 0 \\
\hline
\cdots & -4 & -3 & -2 & -1 & 0 & 1 & \cdots & d\backslash i
\end{array}
$$

in which all entries not shown are zero. The value of this functional on the table T_3 shown above, for example, is $12 - 16 = -4$, proving that no multiple of T_3 can be a cohomology table. Shifting this functional to $L((-1+e, 1+e, 2+e, 3+e), \phi^2(2))$ we get a collection of functionals that prove the corresponding statement for any table between $\gamma(\mathcal{I}_X)$ and $\gamma(\mathcal{I}_Y)$ that has the pattern $0, 1$ somewhere in the h^1 row, *except* for T_2. However, the functional $L((-1, 0, 1, 2, 5), \phi^2(3))$, which is given by the dot product with the table

									3
			12						2
		-12			240	-540	432		1
	12			-240	540	-432	120		0
$\cdots$	-5	-4	-3	-2	-1	0	1	$\cdots$	$d\backslash i$

takes the value $432 - 540 + 12 \cdot 8 = -12$ on T_2, proving the claim.

2 Subtracting Once

As we execute the the Algorithm 0.4 we may leave the class of cohomology tables of coherent sheaves. We will say that a table is *admissible* if it satisfies conditions 1-3 below. As we shall see, the tables produced by the decomposition algorithm will all be admissible.

The first two conditions that an admissible table $\gamma \in \prod \mathbb{R}^{n+1}$ must satisfy are:

1. $\gamma_{i,d} = 0$ for $i > 0$ and $d \gg 0$.

2. The function $d \mapsto \chi_d(\gamma)$ from $\mathbb{Z}$ to $\mathbb{R}$ is a polynomial of degree $s' \leq \dim \gamma$.

We will see that, in fact, admissibility implies that the degree of the polynomial in condition 2 is exactly $\dim \gamma$ (Corollary 2.2.)

For the last condition we need two definitions. Suppose that γ is a table satisfying 1 and 2. Suppose that the dimension of γ is s, and let $z_1 > \cdots > z_s$ be the regularity sequence of γ, as defined above. We call the table positions

$$\{(i, d) \mid z_{i+1} < d + i < z_i\}$$

the *top positions* of γ, and all other positions with possibly nonzero values

$$\{(i, d) \mid d + i \leq z_{i+1}\}$$

the *lower positions* of γ. The last condition for admissibility is:

3. The values at the lower positions of γ coincide with the values of the cohomology table of a coherent sheaf. That is, there exists a coherent sheaf $\mathcal{F}$ such that

$$\gamma_{i,d} = h^i(\mathcal{F}(d)) \text{ for all lower positions } (i, d) \text{ of } \gamma$$

Now let γ be an admissible table of dimension s with regularity sequence $z = z(\gamma) = (z_1, \ldots, z_s)$, for example one whose shape is suggested by Figure 2.

We want to subtract a suitable multiple $q_z \gamma^z$ of a supernatural table γ^z so that, in $\gamma - q_z \gamma^z$, at least one of the corner values becomes zero, and the other corner values remain non-negative. Figures 2 and 3 give an idea of the pattern.

To achieve this goal we must take

$$q_z = \min\{\frac{\alpha_0}{a_0}, \ldots, \frac{\alpha_m}{a_m}\},$$

where $\alpha_0, \ldots, \alpha_m$ and $a_0, \ldots, a_m$ denote the corner values of γ and γ^z respectively. The main step in the proof of Theorem 0.2 is to show that *all* of the entries of $\gamma - q_z \gamma^z$ are non-negative. This is the content of Proposition 2.1.

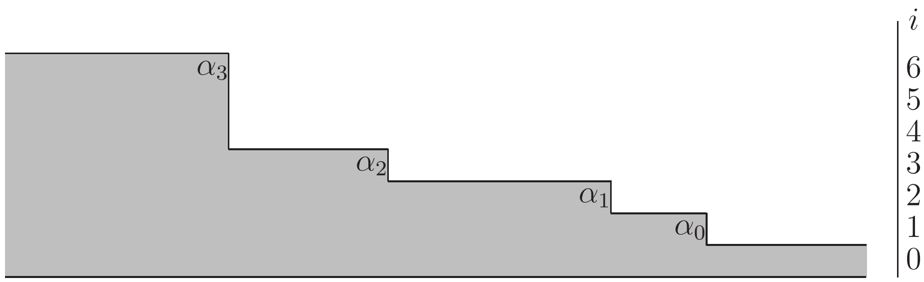

Figure 2: A cohomology table of dimension $s = 6$. The shaded region indicates where the table may have nonzero elements. The α_i are the corner values.

Proposition 2.1. *Let γ be an admissible table of dimension $s > 0$ with regularity sequence $z = (z_1, \dots, z_s)$. Let γ^z be the cohomology table of a supernatural sheaf of dimension $s = \dim \gamma$ with root sequence z. Let*

$$q_z = \min\{\frac{\gamma_{i,z_i+i-1}}{\gamma^z_{i,z_i+i-1}} \mid i \leq s \text{ and } z_{i+1} < z_i - 1\}$$

be the minimal ratio of the corner values of γ and γ^z. Then all entries of the table

$$\gamma - q_z \gamma^z$$

are non-negative, and its regularity sequence is $< z$.

Corollary 2.2. *If γ is a nonzero admissible table, then the function $d \mapsto \chi_d(\gamma)$ is a polynomial of degree exactly $\dim \gamma$.*

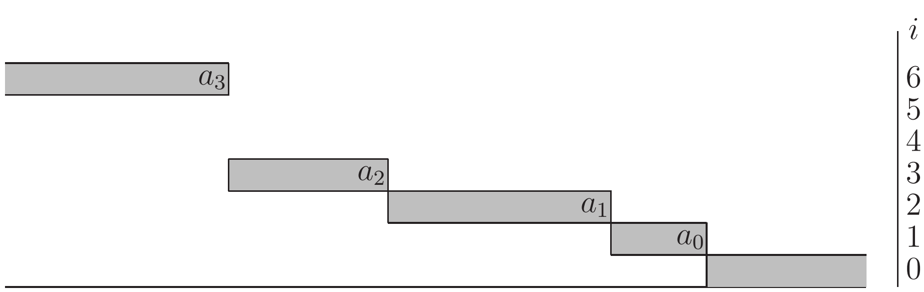

Figure 3: Supernatural table γ^z corresponding to the cohomology table in Figure 2. Here the a_i are the corner values. The grayed area, where this table has nonzero values, coincides with the *top positions* of the table in Figure 2.

Proof of Corollary 2.2. For $d \gg 0$, the entry on the d-th diagonal of the table γ^z is positive. Its value is $\prod_1^s (d - z_i)$, and thus grows as a polynomial of degree $s = \dim \gamma^z = \dim \gamma$. If $d \mapsto \chi_d(\gamma)$ had degree $< \dim \gamma$, then $\gamma - q_z \gamma^z$ would have negative entries in these places, contradicting Proposition 2.1. □

Proof of Proposition 2.1. Let j be a cohomological index and t a degree where $\gamma^z_{j,t} \neq 0$, say $z_{j+1} + j < t < z_j + j$. Let $\beta = \gamma_{j,d}$ and $b = \gamma^z_{j,d}$. We must show that $\beta - q_z b \geq 0$.

If $t = z_j + j - 1$ then we are talking about values at a corner position of γ and γ^z, and the assertion follows immediately from the definition of q_z. Thus we suppose that we are not at a corner position, that is, $z_{j+1} + j < t < z_j + j - 1$.

We first treat the case where $j > 0$. Figure 4 illustrates the situation for $j = 2$.

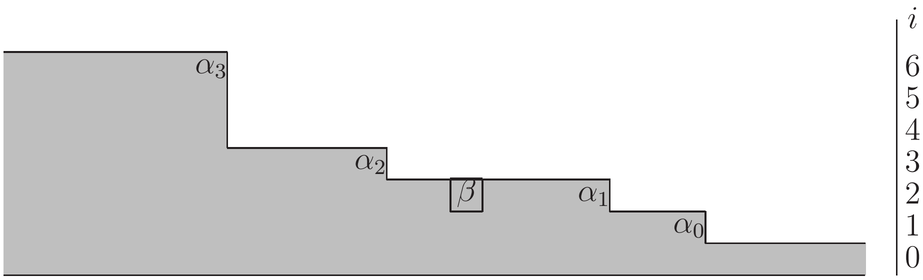

Figure 4: The case $j > 0$ (here $j = 2$). We must prove that the entry $\beta - q_z b$, of the table $\gamma - q_z \gamma^z$, is non-negative. Figure 5 shows the corresponding entry of γ^z.

As indicated in the diagram, there is a corner position of γ and γ_z immediately to the right of the position (j, t), and the values there are $\alpha_i := \gamma_{j, z_j + j - 1}$ and

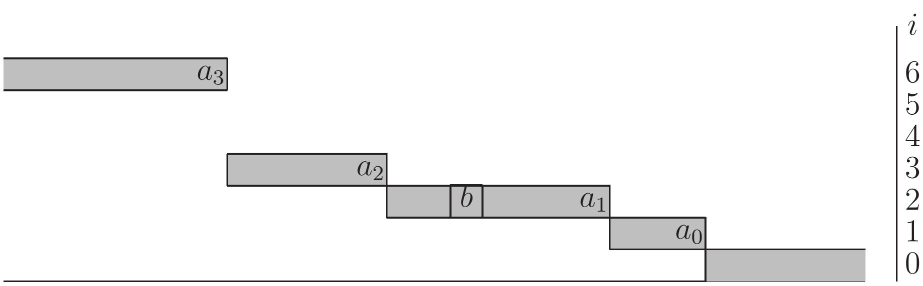

Figure 5: Supernatural table γ^z showing the value b at the same position as that of β in Figure 4.

$a_i := \gamma^z_{j,z_j+j-1}$ respectively. Since $\frac{\alpha_i}{a_i} \geq q_z$ it suffices to prove that

$$\beta - \frac{\alpha_i}{a_i} b \geq 0.$$

To this end, consider the degree sequence

$$d = (d_0, \ldots, d_{s+1}) := (-z_1, \ldots, -z_j, -z_j + 1, -t + j, -z_{j+1}, \ldots, -z_s)$$

and let $r_i = r_i(d)$ as usual. Since $\chi_{z_i}(\gamma^z) = 0$ by construction, Lemma 1.1 applied to the table γ^z gives

$$0 = L(d, \infty)(\gamma^z) = \sum_{i=0}^{s+1} (-1)^i r_i \chi_{-d_i}(\gamma^z) = r_j a_i - r_{j+1} b,$$

so $b/a_i = r_j / r_{j+1}$, and it suffices to show that $r_{j+1}\beta - r_j \alpha_i \geq 0$.

On the other hand, we may apply Lemma 1.1 to the admissible table γ to get

$$0 = L(d, \infty)(\gamma) = \sum_{i=0}^{s+1} (-1)^i r_i \chi_{-d_i}(\gamma) = r_j \alpha_i - r_{j+1}\beta + L(d, \phi^j)(\gamma).$$

By the choice of the degree sequence d, the formula for $L(d, \phi^j)(\gamma)$ involves only values at the lower positions of γ (see Figure 6.)

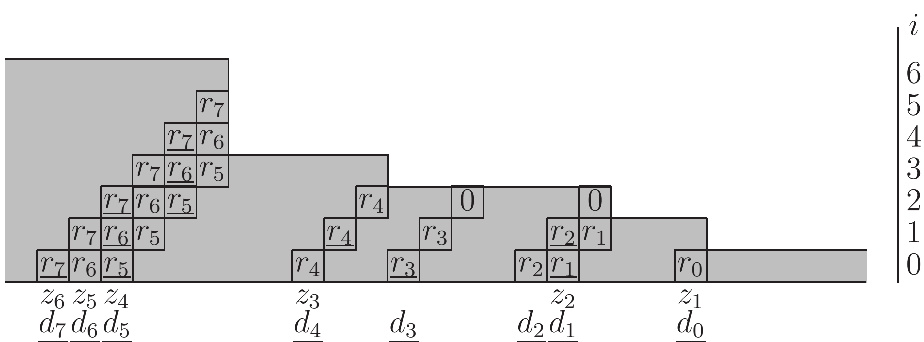

Figure 6: The functional $L(d, \phi^j)$ is the dot product with the table having $\pm r_i$ in the positions shown, and zeros elsewhere. In the illustration, $s = 6$ and $j = 2$. To save space we have denoted $-d_i$ by $\underline{d_i}$ and $-r_i$ by $\underline{r_i}$. The explicit zeros are added for emphasis.

Because γ is admissible, $L(d, \phi^j)(\gamma) = L(d, \phi^j)(\gamma(\mathcal{F}))$ for some coherent sheaf $\mathcal{F}$. Thus we may apply Theorem 1.2 to conclude that

$$r_{j+1}\beta - r_j \alpha_i = L(d, \phi^j)(\gamma) \geq 0$$

as desired.

The proof in the case $j = 0$ is almost the same. Figure 7 Illustrates the position

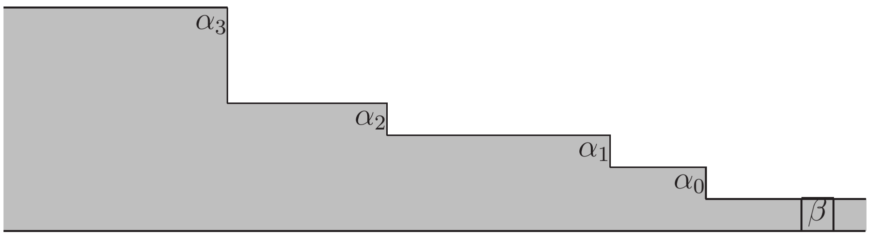

Figure 7: Position of β in case $j = 0$.

of the value β in this case. Since γ^z is assumed nonzero at the position (j, t), we must have $t > z_1$ in this case. This time there is no corner position to the right of $(0, t)$, but we set $i = m$, and we let d be the degree sequence

$$d = (d_0, \ldots, d_{s+1}) = (-t, -z_1, \ldots, -z_s, -z_s + 1).$$

Figure 8 illustrates the relation of γ to the positions involved in the functional $-L(d, \phi^0)$. The rest of the argument is nearly the same.

$$\beta - q_z b \geq \beta - \frac{\alpha_m}{a_m} b \geq 0$$

follows, because

$$0 = L(d, \infty)(\gamma^z) = r_0 b - r_{s+1} a_m$$

gives $b/a_m = r_{s+1}/r_0$, and

$$0 = L(d, \infty)(\gamma) = r_0 \beta - r_{s+1} \alpha_m + L(d, \phi^0)(\gamma)$$

implies the desired positivity, because $-L(d, \phi^0)(\gamma) \geq 0$ by Theorem 1.

$\square$

3 Proof of the main result

We start by describing the growth of dimensions of the cohomology groups $h^i(\mathcal{F}(d))$ for $d \gg 0$.

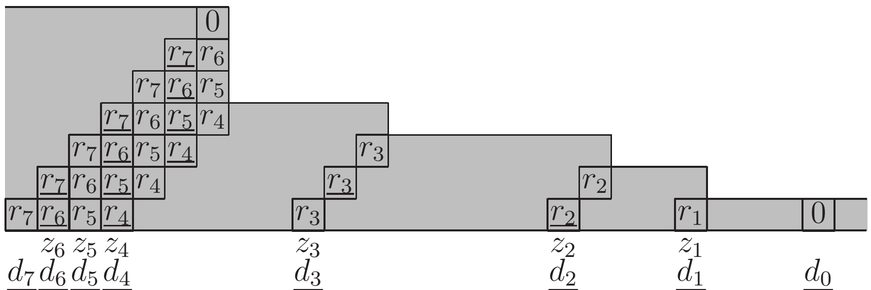

Figure 8: The functional $-L(d, \phi^0)$ is the dot product with the table having $\pm r_i$ in the positions shown, and zeros elsewhere. In the illustration, $s = 6$. To save space we have denoted $-d_i$ by $\underline{d_i}$ and $-r_i$ by $\underline{r_i}$. The explicit zeros are added for emphasis.

Proposition 3.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n$. For each $i = 0, \ldots, n$ there exists a polynomial $p^i_{\mathcal{F}} \in \mathbb{Q}[t]$ such that*

$$p^i_{\mathcal{F}}(d) = h^i(\mathcal{F}(d)) \quad \text{for all} \quad d \ll 0.$$

The degree of $p^i_{\mathcal{F}}$ is $\leq i$, with equality if and only if $\mathcal{F}$ has an associated subvariety of dimension i. In particular, if the dimension of the support of $\mathcal{F}$ is s, then $\deg p^s_{\mathcal{F}} = s$. Furthermore, $\mathcal{F}$ is pure-dimensional, if and only if $\deg p^i_{\mathcal{F}} < s$ for every $i < s$.

Proof. Let M be a graded module over the polynomial ring $S = \mathbb{K}[x_0, \ldots, x_n]$ whose associated sheaf is $\mathcal{F}$. For $i > 0$,

$$\begin{aligned} \oplus_d \operatorname{Hom}_{\mathbb{K}}(H^i(\mathcal{F}(d)), \mathbb{K}) &= \oplus_d \operatorname{Ext}^{n-i}(\mathcal{F}(d), \omega_{\mathbb{P}^n}) \\ &= \oplus_d \operatorname{Ext}^{n-i}(\mathcal{F}, \mathcal{O}(-n-1-d)) \\ &= \operatorname{Ext}^{n-i}_S(M, S(-n-1)). \end{aligned}$$

Thus $p^i_{\mathcal{F}}$ is the Hilbert polynomial of $\operatorname{Ext}^{n-i}_S(M, S(-n-1))$, so the degree of $p^i_{\mathcal{F}}$ is one less than the Krull dimension of $\operatorname{Ext}^{n-i}_S(M, S(-n-1))$, or, equivalently, of $\operatorname{Ext}^{n-i}_S(M, S)$.

The inequality $\deg p^i_{\mathcal{F}} \leq \dim \mathcal{F}$ now follows from the Auslander-Buchsbaum-Serre Theorem: after localizing S at any prime P of dimension $> i+1$ we get a regular local ring of dimension $< (n+1)-(i+1) = n-i$, so $\operatorname{Ext}^{n-i}(M, S)_P = 0$. It follows that $\dim \operatorname{Ext}^{n-i}(M, S) < i+1$. Now suppose that P is a prime of dimension exactly $i+1$. By the Auslander-Buchsbaum formula, P is associated

to M if and only if the projective dimension of M_P is $i+1$, which is true if and only if $\mathrm{Ext}^{n-i}(M,S)_P \neq 0$. Since every associated prime of a graded module is homogeneous, P must correspond in this case to an associated subvariety of $\mathcal{F}$, proving the statement about equality. The rest of the Proposition follows. $\square$

Proof of Theorem 0.2. For the first statement of the Theorem it suffices to show that Algorithm 0.4 succeeds. We have already seen in Proposition 2.1 that Step 3c, starting with an admissible table, always produces a new admissible table, and we have explained in the Outline of the Proof in the Introduction why the dimension of γ will drop by at least 1 each time we reach Step 2. Thus it suffices to show that if we start with an admissible table γ, then the sequence of tables produced by the WHILE loop of Step 3 actually converges to an admissible table, so that we can execute Step 4.

Convergence is no problem: By Proposition 2.1, the tables stay admissible, and thus have only non-negative terms throughout an instance of Step 3. Thus the values in a given position form a decreasing, bounded below sequence.

To show that the limiting table produced in Step 4 is actually admissible, suppose the rows of cohomological index $s'+1,\dots,s$ are wiped out by a pass through Step 3, while the s'-th row remains nonzero. We have to show that the remaining table γ' is an admissible table of dimension s'. It is clear, in any case, that γ' satisfies Condition 1 of admissibility.

Since the rows with cohomological index $0,\dots,s'$ survive, only finitely many corner values with cohomological index $j \leq s'$ are removed in the course of Step 3. So we may replace γ with the admissible table that results from finitely many subtractions, and assume that no corner value with cohomological index $\leq s'$ becomes zero in the infinite sequence of subtractions leading to γ'. It follows that the values of in the lower positions of γ' are the same as those in the corresponding positions of γ; thus condition 3 of admissibility is satisfied.

To complete the proof, we first note that the sequence of Hilbert functions of the tables obtained by the successive subtractions converges decreasingly to a function that takes non-negative real values at all $d \gg 0$. At every finite stage we subtract a polynomial of degree $s+1$, so the $s+1$-st difference function is zero. By continuity, it remains zero in the limit. It follows that $d \mapsto \chi_d(\gamma')$, is a polynomial function.

On the other hand, the values on the top row of γ' at the positions $d \ll 0$ grow at most like a polynomial of degree s' since all values are bounded by the values of the corresponding row of γ. The rows with cohomological degree $i < s'$ have for $d \ll 0$ the values of the original cohomology table of $\mathcal{F}$. By Proposition 3.1, they

grow with negative d as polynomials of degree $< s'$. Thus the Euler characteristic $\chi_d(\gamma)$ is a polynomial in d of degree $\leq s'$; that is, γ' satisfies condition 2 of admissibility. This completes the proof that Algorithm 0.4 succeeds, and produces a decomposition of the desired kind.

To prove uniqueness, suppose that Z and W are both chains of root sequences, and that

$$\gamma(\mathcal{F}) = \sum_{z \in Z} q_z \gamma^z = \sum_{w \in W} r_w \gamma^w$$

with q_z and r_w positive real numbers, where Z is the chain produced by Algorithm 0.4. Since Z, at least, is well ordered, there is a largest element of Z that does not appear in W, or appears with a different coefficient. We may as well subtract the contributions of the terms corresponding to larger elements of Z, which are the same for the two sums, and thus suppose that

$$\gamma = \sum_{z \in Z} q_z \gamma^z = \sum_{w \in W} r_w \gamma^w$$

where γ is an admissible table, and the maximal element $\overline{z} \in Z$ either does not appear in W, or appears with a different coefficient $r_{\overline{z}} \neq q_{\overline{z}}$.

Because the root sequence of Z is the regularity sequence of γ, every $w \in W$ must satisfy $w \leq \overline{z}$. If $\overline{z}$ itself is in W, but $r_{\overline{z}} \neq q_{\overline{z}}$, then $\gamma - r_{\overline{z}}\gamma^{\overline{z}}$ has exactly the same corner positions and regularity sequence as γ. But since W is a chain, at least one of the corner positions of γ is represented with the value zero in every one of the γ^w for $\overline{z} \neq w \in W$, and we see that $\gamma - \sum_{w \in W} r_w \gamma^w \neq 0$, contradicting our hypothesis.

Similarly, if $\overline{z} \notin W$ then, since there are only finitely many elements just below z in the poset of root sequences, there is some corner position of γ that is represented by the value zero in every γ^w for $w \in W$, so we can finish the argument in the same way. This proves uniqueness.

Note that the coefficients q_z involved in any finite sequence of subtractions in Algorithm 0.4 starting from a rational cohomology table are automatically rational. This applies to all the q_z corresponding to γ^z of dimension $= \dim \mathcal{F}$.

Now suppose that $\mathcal{F}$ is a pure-dimensional sheaf. It suffices to show that the decomposition is obtained as the limit of finite sequences of subtractions starting from the cohomology table of $\mathcal{F}$ in this case.

Once again, let γ' be the result of subtracting the cohomology tables of vector bundles on $\mathbb{P}^s$, as in Algorithm 0.4, so that $s' := \dim \gamma' < \dim \gamma(\mathcal{F}) = s$. By Proposition 3.1, the Euler characteristic of the resulting table γ' grows like

a polynomial of degree $\leq s'-1$. If γ' were nonzero, we would get a contradiction to Corollary 2.2. Thus $\gamma' = 0$, completing the proof. $\square$

4 Proof of the Positivity Theorem

In our paper [2009] we defined pairings

$$\langle \beta, \gamma \rangle = \sum_{\{i,j,k \mid j \leq i\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k}.$$

and

$$\begin{aligned} \langle \beta, \gamma \rangle_{c,\tau} &= \sum_{\{i,j,k \mid j \leq i \text{ and } (j<\tau \text{ or } j \leq i-2)\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k} \\ &+ \sum_{\{i,j,k,\epsilon \mid 0 \leq \epsilon \leq 1,\ j=\tau,\ i=j+\epsilon,\ k \leq c+\epsilon\}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k}. \end{aligned}$$

for $\beta = (\beta_{i,k}) \in \oplus_{-\infty}^{\infty} \mathbb{R}^{n+2}$ and $\gamma \in \prod_{-\infty}^{\infty} \mathbb{R}^{n+2}$, and $0 \leq \tau \leq n, c \in \mathbb{Z}$. We showed that, if β is the Betti table of a finitely generated graded module over $S := \mathbb{K}[x_0, \ldots, x_n]$ and γ is the cohomology table of a vector bundle $\mathcal{F}$, or of a complex E of free graded S-modules, supported in positive cohomological degrees, then

$$\langle \beta, \gamma \rangle \geq 0 \quad \text{and} \quad \langle \beta, \gamma \rangle_{c,\tau} \geq 0.$$

Our proof for the vector bundle case reduced to the case of a free complex by replacing the vector bundle with a free monad. Since the free monads of coherent sheaves have terms in negative cohomological degrees, this proof could not show that the pairing above was non-negative when $\mathcal{F}$ is a general coherent sheaf. After our paper was finished, Rob Lazarsfeld pointed out to us a variation on our proof in which the monad for $\mathcal{F}$ is replaced by an injective or flasque resolution of $\mathcal{F}$. It turns out that, with one further idea, this idea yields a proof of non-negativity that works for any coherent sheaf $\mathcal{F}$.

Theorem 4.1. *Let F be the minimal free resolution of a finitely generated graded S-module M. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^n$, then*

$$\langle F, \mathcal{F} \rangle \geq 0 \quad \textit{and} \quad \langle F, \mathcal{F} \rangle_{c,\tau} \geq 0$$

Proof. The number $\langle F, \mathcal{F}\rangle$ depends only on the dimensions of the $H^j(\mathcal{F}(-k))$ for $k \in \mathbb{Z}$, we may begin by replacing $\mathcal{F}$ with a "general translate" by an element of $PGL(n)$, to make $\mathcal{F}$ homologically transverse to the sheaf $\tilde{M}$, as proven by Sierra [2007] and by Miller and Speyer [2008]. If we let G be a graded S-module such that $\widetilde{G} = \mathcal{F}$, this means that the modules $\mathrm{Tor}_i^S(M, G)$ have support only at the irrelevant ideal for $i > 0$.

Let $E : \oplus_\ell G[x_\ell^{-1}] \to \cdots$ be the Čech complex of G. The homological transversality implies that the complex $F \otimes E^j$ has homology only at $F_0 \otimes E^j$, so the total complex of the double complex $F \otimes E$ has homology only in non-negative cohomological degree. We can now proceed exactly as in the proofs of Theorems 3.1 and 4.1 of our [2009]. ☐

We next describe a simplification in the statement that makes use of the main results of our [2009] and of Boij-Söderberg [2008], and also an extension of the statement that will be crucial for the proof of Theorem 0.2.

Recall that a graded Cohen-Macaulay S-module M of codimension $s+1$ is said to have a *pure* resolution with degree sequence $d = (d_0, \ldots, d_{s+1})$ if the minimal free resolution of M has the form

$$S(-d_0)^{r_0} \longleftarrow S(-d_1)^{r_1} \longleftarrow \cdots \longleftarrow S(-d_{s+1})^{r_{s+1}} \longleftarrow 0.$$

In this case, $d_0 < \cdots < d_{s+1}$, and there is a positive rational number q such that each $r_i = q \cdot r_i(d)$, where, as in §4

$$r_i(d) := \prod_{\substack{1 \le j < k \le s+1 \\ j,k \ne i}} (d_k - d_j)$$

(See Herzog and Kühl [1984]).

Together, our [2009] and Boij-Söderberg [2008] show that there is a graded Cohen-Macaulay S-module with any given degree sequence $(d_0 < \cdots < d_{s+1})$, and the Betti table of any graded S-module is a positive rational linear combination of the Betti tables of Cohen-Macaulay modules with pure resolutions. Thus to prove that the value of a bilinear functional such as those above is non-negative, it suffices to treat the case where β is the Betti table of a Cohen-Macaulay module with pure resolution, and if the resolution has degree sequence d, one may as well assume that $r_i = r_i(d)$ for every i as well: that is, we may restrict our attention to the functionals $\langle(\beta^d, \gamma\rangle_{c,\tau}$ with β^d to be the table with

$$\beta^d : \quad \beta_{i,j} = \begin{cases} r_i(d) & \text{if } j = d_i \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For such β^d we may re-write the definition given above in the form:

$$\begin{aligned}\langle \beta^d, \gamma\rangle_{c,\tau} = \sum_{i<\tau} &(-1)^i\, r_i(d)\, \chi^{\leq i}_{-d_i}(\gamma) \\ &+(-1)^\tau\, r_\tau(d)\, \chi^{\leq A}_{-d_\tau}(\gamma) \\ &+(-1)^{\tau+1} r_{\tau+1}(d)\, \chi^{\leq B}_{-d_{\tau+1}}(\gamma) \\ + \sum_{i>\tau+1} &(-1)^i\, r_i(d)\, \chi^{\leq i-2}_{-d_i}(\gamma)\end{aligned}$$

where

$$A = \begin{cases} \tau - 1 & \text{if } c < d_\tau \\ \tau & \text{otherwise} \end{cases}$$

$$B = \begin{cases} \tau - 1 & \text{if } c < d_{\tau+1} \\ \tau & \text{otherwise} \end{cases}$$

It follows that if $\tau \geq 1$ and $c < d_\tau$ then

$$\langle \beta^d, \gamma\rangle_{c,\tau} = L(d, \phi^\tau)(\gamma)$$

while if $c \geq d_{\tau+1}$ then

$$\langle \beta^d, \gamma\rangle_{c,\tau} = L(d, \phi^{\tau+1})(\gamma).$$

Moreover, if the $\gamma_{i,j}$ are non-negative, as in any admissible table, and $d_\tau \leq c < d_{\tau+1}$ then, comparing signs, we see that

$$\langle \beta^d, \gamma\rangle_{c,\tau} \geq \langle \beta^d, \gamma\rangle_{d_\tau - 1,\tau} = L(d, \phi^\tau)(\gamma)$$

so this case is not very useful.

Proof of Theorem 1.2. The description above shows that the cases $j > 0$ follow from Theorem 4.1.

To simplify the notation for the case $j = 0$ we set $\psi = \phi^0 = (-1, 0, 1, \ldots, s-2, s-1, s-1)$. We write

$$d^{(j)} = (d_1, \ldots, d_j) \qquad \text{for } j = 1, \ldots, s+1$$

and

$$\psi^{(j)} = \begin{cases} (0, 1, \ldots, j-1) & \text{for } j = 1, \ldots, s, \text{ and} \\ (0, 1, \ldots, s-1, s-1) & \text{for } j = s+1. \end{cases}$$

We will show that

$$(1)\quad -L(d,\psi) = \sum_{\ell=0}^{s+1} (-1)^{s-\ell} r_{s+1-\ell}(d)\chi_{-d_{s+1-\ell}}^{\leq \psi_{s+1-\ell}} = \sum_{k=0}^{s} A_k\ L(d^{(s+1-k)}, \psi^{(s+1-k)})$$

where

$$A_k = \prod_{1\leq j\leq s-k} (d_j - d_0) \prod_{\substack{1\leq i<j\leq s+1\\ s+1-k<j}} (d_j - d_i).$$

The coefficients A_k are obviously non-negative. By Theorem 4.1, the forms $L(d^{(s+1-k)}, \psi^{(s+1-k)})$ take non-negative values on the cohomology tables of coherent sheaves, so this will suffice to prove Theorem 1.2.

The coefficient of $(-1)^{s-\ell}\chi_{-d_{s+1-\ell}}^{\leq \psi_{s+1-\ell}}$ on the right-hand side of Equation (1) is

$$\sum_{k=0}^{\ell} \Bigg(\prod_{1\leq j\leq s-k} (d_j - d_0)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1\\ s+1-k<j}} (d_j - d_i)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1-k\\ i,j\neq s+1-\ell}} (d_j - d_i)\Bigg).$$

We will show that this is $r_{s+1-\ell}(d)$. The terms in the sum have a common factor (coming from the first and third factors in each term)

$$\Bigg(\prod_{1\leq j\leq s-\ell} (d_j - d_0)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1-\ell\\ i,j\neq s+1-\ell}} (d_j - d_i)\Bigg) = \prod_{\substack{0\leq i<j\leq s+1-\ell\\ i,j\neq s+1-\ell}} (d_j - d_i).$$

After factoring this out, we get

$$\sum_{k=0}^{\ell} \Bigg(\prod_{s-\ell+1\leq j\leq s-k} (d_j - d_0)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1\\ s+1-k<j}} (d_j - d_i)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1\\ i,j\neq s+1-\ell\\ s+1-\ell<j\leq s+1-k}} (d_j - d_i)\Bigg),$$

which can be further factored as

$$\Bigg(\prod_{\substack{1\leq i<j\leq s+1\\ i,j\neq s+1-\ell\\ s+1-\ell<j}} (d_j - d_i)\Bigg) \sum_{k=0}^{\ell} \Bigg(\prod_{s-\ell+1\leq j\leq s-k} (d_j - d_0)\Bigg)\Bigg(\prod_{\substack{i=s+1-\ell\\ s-k+1<j}} (d_j - d_i)\Bigg).$$

Applying the case $t=-1$ of Lemma 4.2, we can combine all the factors to express the original sum as

$$\Bigg(\prod_{\substack{0\leq i<j\leq s+1-\ell\\ i,j\neq s+1-\ell}} (d_j - d_i)\Bigg)\Bigg(\prod_{\substack{1\leq i<j\leq s+1\\ i,j\neq s+1-\ell\\ s+1-\ell<j}} (d_j - d_i)\Bigg)\Bigg(\prod_{s-\ell+1<j\leq s+1} (d_j - d_0)\Bigg)$$

$$= \prod_{\substack{0 \leq i < j \leq s+1 \\ i,j \neq s+1-\ell}} (d_j - d_i) = r_{s+1-\ell}(d),$$

completing the proof. □

Lemma 4.2. *For* $-1 \leq t \leq \ell - 1$ *we have*

$$\sum_{k=0}^{\ell} \Bigg(\prod_{s-\ell+1 \leq j \leq s-k} (d_j - d_0) \Bigg) \Bigg(\prod_{s-k+1 < j} (d_j - d_{s-\ell+1}) \Bigg)$$

$$= \Bigg(\prod_{s-\ell+1 < j \leq s-t} (d_j - d_0) \Bigg) \Bigg(\prod_{s-t < j} (d_j - d_{s-\ell+1}) \Bigg)$$

$$+ \sum_{k=0}^{t} \Bigg(\prod_{s-\ell+1 \leq j \leq s-k} (d_j - d_0) \Bigg) \Bigg(\prod_{s-k+1 < j} (d_j - d_{s-\ell+1}) \Bigg).$$

Proof. The formula is obvious for $t = \ell - 1$, so we do descending induction. The induction step follows by combining the first product with the $k = t$ term of the summation, as follows:

$$\prod_{j=s-\ell+2}^{s-t} (d_j - d_0) \prod_{s-t<j} (d_j - d_{s-\ell+1}) + \prod_{j=s-\ell+1}^{s-t} (d_j - d_0) \prod_{s-t+1<j} (d_j - d_{s-\ell+1})$$

$$= \Bigg(\prod_{j=s-\ell+2}^{s-t} (d_j - d_0) \Bigg) \Big((d_{s-t+1} - d_{s-\ell+1}) + (d_{s-\ell+1} - d_0) \Big) \Bigg(\prod_{s-(t-1)<j} (d_j - d_{s-\ell+1}) \Bigg)$$

$$= \Bigg(\prod_{j=s-\ell+2}^{s-(t-1)} (d_j - d_0) \Bigg) \Bigg(\prod_{s-(t-1)<j} (d_j - d_{s-\ell+1}) \Bigg).$$

□

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